CCSF PHYC 4D Lecture Notes

Karl M. Westerberg

Inverse Square Forces

Polar coordinates

Points in the x-y plane can be described either in terms of Cartesian coordinates (x, y) or polar coordinates (r, θ). The two sets of coordinates are related according to

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2} \qquad \tan\theta = y/x \tag{1}$$

The following polar unit vectors are useful:

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \qquad \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \tag{2}$$

They are perpendicular to each other and satisfy

$$\frac{d\hat{r}}{d\theta} = \hat{\theta} \qquad \frac{d\hat{\theta}}{d\theta} = -\hat{r} \tag{3}$$

They can be used to resolve any vector into radial and tangential components, respectively.

Conic sections

The easiest way to define a conic section in polar coordinates is to place the *focus* (a point) at the origin and the *directrix* (a line) at $x = d > 0$. The conic section is then defined to be the set of all points whose distance to the focus is equal to the *eccentricity* (denoted e, a non-negative constant) multiplied by the distance to the directrix.

Put into symbols, this translates into

$$r = e|d - r\cos\theta| = \pm e(d - r\cos\theta)$$

For points that are on the near side of the directrix (i.e., same side as the focus), we take the plus sign and solve for r to get

$$r = \frac{ed}{1 + e\cos\theta} \tag{4}$$

For points on the far side of the directrix, we take the minus sign and solve for r to get

$$r = \frac{ed}{-1 + e\cos\theta} \tag{5}$$

Since $r \geq 0$, this latter case can only apply if $e > 1$. In all cases, the minimum allowed value of r occurs at $\theta = 0$ and is given by

$$r_{\min} = \frac{ed}{\pm 1 + e} \tag{6}$$

The eccentricity is used to classify the shape of the conic section.

If $0 < e < 1$, the conic section is an ellipse. It is evident from eq. 4 that the ellipse is bounded: its maximum allowed r occurs at $\theta = \pi$ and is given by

$$r_{\max} = \frac{ed}{1-e} \tag{7}$$

Taking $e \to 0$ and $d \to \infty$ simultaneously so that $r_0 = ed$ remains constant reduces the ellipse to a circle of radius r_0 (eq. 4 reduces to $r = r_0$). Values of e closer to 1 define ellipses that are more flattened.

If $e > 1$, the conic section is a hyperbola. It actually comes in two sections; the near section is described by eq. 4 and the far section is described by eq. 5. It is evident that hyperbolas are unbounded; $r \to \infty$ when $\theta \to \pm\theta_0$, where

$$\theta_0 = \arccos(\mp 1/e) \tag{8}$$

where the minus sign applies for the near section and the plus sign applies for the far section (the two values of θ_0 are supplementary to each other). In either case, the hyperbola is only defined for $-\theta_0 < \theta < \theta_0$. The four values of $\pm\arccos(\mp 1/e)$ define asymptotic directions for the hyperbola.

As e becomes larger, the hyperbola sections get closer to each other and closer to the directrix (the asymptotic directions become closer to $\pm\pi/2$). In the limit $e \to \infty$, both hyperbola sections coincide with the directrix.

If $e = 1$, the conic section is a parabola. Although the parabola is unbounded, it has no asymptotic direction. The parabola is defined for all values of θ except $\theta = \pi$.

Central forces

A central force is a force that is radially directed away from or towards some prescribed force center, and whose radial component depends only on the distance to the force center. The motion of a particle acted upon by a central force is most easily described using polar coordinates, with the force center at the origin. Such a particle has an acceleration given by

$$\vec{a} = f(r)\,\hat{r} \tag{9}$$

where $f(r)$ represents the radial component of the force divided by m, the mass of the particle. Note that $f(r)$ depends only on r, not on θ.

Central forces are conservative forces. The associated potential energy is given by $m\psi(r)$, where $f(r) = -\psi'(r)$. Total mechanical energy is conserved during the motion of the particle, so that

$$\epsilon = E_{\mathrm{mech}}/m = \tfrac{1}{2}|\vec{v}|^2 + \psi(r) \tag{10}$$

is constant.

Central forces also do not exert any torque around the force center (the lever arm is zero), and so angular momentum is also conserved. Thus

$$h = L_z/m = (\vec{r} \times \vec{v})_z \tag{11}$$

is also constant.

Position as a function of time can be written

$$\vec{r}(t) = r(t)\,\hat{r} \tag{12}$$

The velocity and acceleration can be computed by taking time derivatives, keeping in mind that $\hat{r}$ and $\hat{\theta}$ vary as θ varies over time. With the help of eq. 3, we find that

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\,\hat{\theta} \qquad \frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\,\hat{r} \tag{13}$$

The notation $\dot{q}$ is short-hand for dq/dt for any time-dependent quantity q. The velocity and acceleration can now be calculated.

$$\begin{aligned} \vec{v} &= \dot{\vec{r}} = \dot{r}\,\hat{r} + r\,\dot{\hat{r}} \\ &= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} \end{aligned} \tag{14}$$

$$\begin{aligned} \vec{a} &= \dot{\vec{v}} = \ddot{r}\,\hat{r} + \dot{r}\dot{\theta}\,\hat{\theta} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}^2(-\hat{r}) \\ &= (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\theta} \end{aligned} \tag{15}$$

The value of h can now be evaluated in terms of r and θ.

$$h = (\vec{r} \times \vec{v})_z = (r)(v_\theta) = r^2\dot{\theta} \tag{16}$$

Its time derivative is given by

$$\dot{h} = d(r^2\dot{\theta})/dt = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r a_\theta$$

For a central force, $a_\theta = 0$, and so h is indeed a constant as we claimed earlier.

Setting $a_r = f(r)$ allows us to solve (at least in principle) for $r(t)$ by solving a second-order differential equation. Eq. 16 can be used to eliminate $\dot{\theta}$.

$$\ddot{r} - r\dot{\theta}^2 = \ddot{r} - h^2/r^3 = f(r) \tag{17}$$

After solving for $r(t)$, it is then possible to solve for $\theta(t)$ by integration:

$$\theta(t) = \theta(0) + \int_0^t \dot{\theta}(t')\,dt' = \theta(0) + \int_0^t h/r(t')^2\,dt'$$

The radial motion will appear one dimensional to an observer rotating with the particle. The term $-h^2/r^3$ $(= -r\dot{\theta}^2 = -r\omega^2)$ in eq. 17 represents the centrifugal acceleration measured in that frame.

The value of ϵ can also be evaluated in terms of r and θ.

$$\epsilon = \tfrac{1}{2}|\vec{v}|^2 + \psi(r) = \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}r^2\dot{\theta}^2 + \psi(r) = \tfrac{1}{2}\dot{r}^2 + h^2/(2r^2) + \psi(r) \tag{18}$$

Its time derivative is given by

$$\dot{\epsilon} = \dot{r}\ddot{r} + (-2)h^2/(2r^3)\dot{r} + \psi'(r)\dot{r} = \dot{r}(\ddot{r} - h^2/r^3 - f(r)) = 0$$

according to eq. 17. Evidently, ϵ is also a constant, as we had claimed earlier.

Inverse square forces

An inverse square force is characterized by

$$\vec{a} = f(r)\,\hat{r} = \frac{A}{r^2}\,\hat{r} \tag{19}$$

The force may either be attractive $(A < 0)$ or repulsive $(A > 0)$. The corresponding potential function is given by $\psi(r) = A/r$. Both $h = r^2\dot{\theta}$ and $\epsilon = \frac{1}{2}\dot{r}^2 + h^2/(2r^2) + A/r$ are constants of the motion.

The differential equation for $r(t)$ is given by

$$\ddot{r} - h^2/r^3 = A/r^2 \tag{20}$$

Rather than try to solve r directly in terms of t, we will attempt to determine the path followed by the particle by solving r in terms of θ.

We define $u = 1/r$ in the hopes that this simplies the differential equation. Far from being a shot in the dark, we might have reason to believe this would happen since $u(\theta)$ is expected to be the sum of a cosine term plus a constant if the motion were to follow a conic section. This is indeed what happens.

To set this up, we first evaluate $\dot{r}$ and then $\ddot{r}$ in terms of u and its derivatives with respect to θ (first and second derivatives denoted u' and u'', respectively).

$$\begin{aligned}
\dot{r} &= \frac{d(1/u)}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -r^2\dot{\theta}u' = -hu' \\
\ddot{r} &= \frac{d}{dt}(-hu') = -h\frac{d(u')}{d\theta}\frac{d\theta}{dt} = -h\dot{\theta}u'' = -h^2u^2u''
\end{aligned} \tag{21}$$

Plugging this into eq. 20 gives

$$-h^2u^2u'' - h^2u^3 = Au^2$$

Dividing out $-h^2u^2$ yields

$$u'' + u = -A/h^2 \tag{22}$$

This is a linear second-order differential equation with constant coefficients. The general solution is given by

$$u = C\cos(\theta - \theta_0) - A/h^2$$

Setting $\theta_0 = 0$ and imposing $C > 0$ (arbitrary choice of coordinate system) yields

$$\frac{1}{r} = u = C\cos\theta - A/h^2 \tag{23}$$

which can also be written as

$$r = \frac{h^2/|A|}{-\operatorname{sign}(A) + (Ch^2/|A|)\cos\theta} \tag{24}$$

where $\operatorname{sign}(A) = \pm 1$ according to whether $A > 0$ or $A < 0$. This is indeed the equation for a conic section, with $e = Ch^2/|A|$ and $d = 1/C$. Imposing the conditions $C > 0$ and $\theta_0 = 0$ ensures that the point of closest approach (smallest r) will occur at $\theta = 0$.

Evaluating ϵ in terms of this solution will allow us to solve for C in terms of A, h, and ϵ.

$$\begin{aligned}
\epsilon &= \tfrac{1}{2}\dot{r}^2 + h^2/(2r^2) + A/r = \tfrac{1}{2}(-hu')^2 + \tfrac{1}{2}h^2u^2 + Au \\
&= \tfrac{1}{2}h^2(-C\sin\theta)^2 + \tfrac{1}{2}h^2(C\cos\theta - A/h^2)^2 + A(C\cos\theta - A/h^2) \\
&= \tfrac{1}{2}h^2C^2\sin^2\theta + \tfrac{1}{2}h^2C^2\cos^2\theta - h^2C(A/h^2)\cos\theta + \tfrac{1}{2}h^2A^2/h^4 + AC\cos\theta - A^2/h^2 \\
&= \tfrac{1}{2}h^2C^2 - \tfrac{1}{2}A^2/h^2
\end{aligned}$$

Note that ϵ is independent of θ, as expected. Solving for C (taking the positive root) yields

$$C = \sqrt{\frac{A^2}{h^4} + \frac{2\epsilon}{h^2}} \tag{25}$$

The eccentricity of the orbit can now be written as

$$e = \frac{Ch^2}{|A|} = \sqrt{1 + \frac{2\epsilon h^2}{A^2}} \tag{26}$$

Elliptical and circular orbits

Elliptical and circular orbits have $e < 1$ and are bound orbits. It is evident from eq. 26 that $e < 1$ is equivalent to $\epsilon < 0$ — bound orbits have a negative mechanical energy. This should make physical sense: the orbitting body is unable to reach $r \to \infty$ because $\epsilon < \psi(\infty) = 0$. Since eq. 5 is not valid for $e < 1$, it follows from eq. 24 that bound orbits can only occur when $A < 0$. This should also make physical sense: negative mechanical energy is only possible if the inverse square force is attractive.

Since $e^2 = 1 + 2\epsilon h^2/A^2 \geq 0$, it follows that

$$\epsilon \geq -\frac{A^2}{2h^2} \tag{27}$$

The minimum value of mechanical energy occurs when $e = 0$. It makes sense that bound orbits over time trend towards circular orbits when frictional losses (due to tidal forces) are taken into account.

Kepler proposed three laws of planetary motion.

1. The orbit of every planet around a sun is an ellipse with the sun at one of its foci.
2. The line joining a planet and the sun sweeps out equal areas during equal time intervals.
3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Kepler based his laws on observational data. These laws can be proved from the fact that the gravitational force is an inverse square force (with $A = -GM_{\text{sun}}$). We have already established the first law.

To prove the second law, we first need to establish a formula for the area swept out by the radial line joining the planet and sun during a short time interval dt. If the planet moves in a circular orbit (r is constant), then the area swept is simply the area of circular arc of angle $d\theta$

$$d(\text{Area}) = \tfrac{1}{2}r^2\,|d\theta| = \tfrac{1}{2}|h|\,dt \tag{28}$$

The latter equality follows from $h = r^2\dot{\theta}$. If the motion is not circular (so that r changes as well), then one can easily see that

$$\tfrac{1}{2}r_<^2\,|d\theta| \leq d(\text{Area}) \leq \tfrac{1}{2}r_>^2\,|d\theta|$$

where $r_<$ and $r_>$ represent the minimum and maximum values of r during this short time interval. As $dt \to 0$, both $r_<$ and $r_>$ approach r, demonstrating that eq. 28 applies in this case as well. Since h is a constant, equal areas must be swept during equal time intervals, at least in the limit as $dt \to 0$. For an extended time interval, one simply integrates both sides of eq. 28 to reach the same conclusion. Note that this law is equivalent to conservation of angular momentum, and applies to any central force law. The force need not be inverse square.

To prove the third law (and derive the proportionality constant), we start by integrating eq. 28 over a complete period.

$$\pi ab = \tfrac{1}{2}|h|T \tag{29}$$

where T is the period of the orbit, a is the semi-major axis, and b is the semi-minor axis of the ellipse (πab is the area of the ellipse). The semi-major axis is given by

$$a = \tfrac{1}{2}(r_{\min} + r_{\max}) = \tfrac{1}{2}ed\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{ed}{1-e^2} \tag{30}$$

The y-coordinate of a point on the ellipse is given by

$$r\sin\theta = \frac{ed\sin\theta}{1+e\cos\theta} \tag{31}$$

Setting the θ derivative of eq. 31 to zero yields $\cos\theta = -e$ (this requires some effort), and thus $\sin\theta = \sqrt{1-e^2}$. Plugging that back into eq. 31 yields the maximum possible y-coordinate, which is the semi-minor axis:

$$b = \frac{ed\sqrt{1-e^2}}{1-e^2} = \frac{ed}{\sqrt{1-e^2}} = a\sqrt{1-e^2} \tag{32}$$

Now solve eq. 29 for T and plug in eqs. 30 and 32 (and $ed = h^2/|A|$ from eq. 24) to obtain

$$\begin{aligned} T &= \frac{2\pi ab}{|h|} = \frac{2\pi}{|h|}\left(\frac{ed}{1-e^2}\right)\left(\frac{ed}{\sqrt{1-e^2}}\right) = \frac{2\pi}{|h|}\frac{(ed)^2}{(1-e^2)^{3/2}} \\ &= \frac{2\pi\sqrt{ed}}{|h|}\left(\frac{ed}{1-e^2}\right)^{3/2} = \frac{2\pi\sqrt{h^2/|A|}}{|h|}a^{3/2} = \frac{2\pi}{\sqrt{|A|}}a^{3/2} \end{aligned} \tag{33}$$

For planetary motion, $|A| = GM_{\text{sun}}$, which is independent of which planet we are considering.

Parabolic and hyperbolic orbits

Parabolic and hyperbolic orbits have $e \geq 1$ and are unbound orbits.

For a parabolic orbit ($e = 1$), it is evident from eq. 26 that $\epsilon = 0$ — parabolic orbits have zero total mechanical energy. Since eq. 5 is not valid for $e = 1$, it follows from eq. 24 that parabolic orbits can only occur when $A < 0$. The parabolic orbit is the boundary case between the elliptical ($e < 1$) bound orbits discussed in the last section and the hyperbolic ($e > 1$) unbound orbits to be discussed later in this section. Since mechanical energy is zero, any kinetic energy that the particle might have must be balanced exactly by a negative potential energy. As the particle approaches $r \to \infty$, the speed of the particle must approach zero. If the particle loses even the slightest amount of mechanical energy to friction, it will no longer be able to escape to infinity, and will instead come back, following an elliptical orbit.

Hyperbolic orbits have positive total mechanical energy ($\epsilon > 0$) and can exist for either attractive ($A < 0$) or repulsive ($A > 0$) inverse square forces. In the case of an attractive

force, the particle (at finite radius) has a negative potential energy, but that is more than made up for by a positive kinetic energy — the particle has enough kinetic energy to escape the attracting force center. In the case of a repulsive inverse square force, both potential energy and kinetic energy are positive; the path *must* be hyperbolic in this case.

A hyperbola has two sections. The section closer to the focus (on the same side of the directrix) is described by eq. 4, and represents the path of the particle if the force is attractive ($A < 0$). The section farther from the focus (on the opposite side of the directrix) is described by eq. 5, and represents the path of the particle if the force is repulsive ($A > 0$). Note that the path bends *towards* the force center for $A < 0$ and *away from* the force center for $A > 0$. This should make sense.

When the particle is very far away ($r \to \infty$), either before or after its one interaction with the force center, it is essentially moving in a straight line at constant speed. Let v_0 represent that speed, and let b represent the perpendicular distance between the force center and the linear trajectory of the particle far away from the force center. This distance is called the *impact parameter* and would represent the distance of closest approach to the force center *if the particle were to continue to move in a straight line as it approaches the force center.* Both ϵ and h can be expressed in terms of these quantities:

$$\epsilon = (K+U)/m = K_\infty/m = \tfrac{1}{2}v_0^2 \tag{34}$$

$$|h| = |\vec{r}_\infty \times \vec{p}_\infty|/m = bv_0 \tag{35}$$

Note that since both ϵ and h are conserved throughout the motion, the speed and impact parameter of the particle will be the same both long before and long after the interaction takes place. The eccentricity of the orbit is given by

$$e = \sqrt{1 + \frac{2\epsilon h^2}{A^2}} = \sqrt{1 + \frac{b^2 v_0^4}{A^2}} \tag{36}$$

Note that e increases (implying a straighter path) as v_0 increases, as b increases, and/or as $|A|$ decreases.

One quantity of particular interest is the scattering angle ϕ, defined to be the angle between the initial and final trajectories of the particle as it approaches and then eventually recedes from the force center. As $r \to \infty$, the polar angle $\theta \to \pm\theta_0$, which is given by eq. 8

$$\cos(\theta_0) = \mp\frac{1}{e} = \frac{\mp 1}{\sqrt{1 + 2\epsilon h^2/A^2}} \tag{37}$$

If $A > 0$, we use the plus sign (far section of the hyperbola) and note that θ_0 is an acute angle of the right triangle whose sides are given by

$$1 \text{ (adjacent)} \qquad \frac{\sqrt{2\epsilon}|h|}{|A|} \text{ (opposite)} \qquad \sqrt{1 + \frac{2\epsilon h^2}{A^2}} \text{ (hypoteneus)} \tag{38}$$

The Pythagorean Theorem is used to compute the opposite side. It follows that

$$\tan(\theta_0) = \frac{\sqrt{2\epsilon}|h|}{A} = \frac{bv_0^2}{A} \tag{39}$$

Since θ changes from $-\theta_0$ to $+\theta_0$ (or vice versa if $h < 0$), it follows that $\phi = \pi - 2\theta_0$, and so

$$\cot(\tfrac{1}{2}\phi) = \frac{\sqrt{2\epsilon}|h|}{|A|} = \frac{bv_0^2}{|A|} \tag{40}$$

If $A < 0$, then we use the minus sign from eq. 37 (near section of the hyperbola) and note that this time, $\pi - \theta_0$ is an acute angle of the right triangle whose sides are given by eq. 38. It follows that θ_0 itself satisfies eq. 39 (note the lack of absolute values on A in that equation — $\tan(\theta_0) < 0$ in this case). This time the scattering angle is given by $\phi = 2\theta_0 - \pi$, and so eq. 40 holds in this case as well (note the absolute values on A; the scattering angle is always defined between 0 and π, and so $\cot(\frac{1}{2}\phi) > 0$).

The scattering angle in either case is given by eq. 40. Solving for b yields

$$b = \frac{|A|}{v_0^2}\cot(\tfrac{1}{2}\phi) = b_0 \cot(\tfrac{1}{2}\phi) \tag{41}$$

where $b_0 = |A|/v_0^2$ is the impact parameter for 90° scattering. This result forms the basis of the Rutherford scattering formula, along with any other scattering experiment involving an inverse square force.